

# The spectral excess theorem for distance-regular graphs having distance- $d$ graph with fewer distinct eigenvalues

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## Abstract

Let  $\Gamma$  be a distance-regular graph with diameter  $d$  and Kneser graph  $K = \Gamma_d$ , the distance- $d$  graph of  $\Gamma$ . We say that  $\Gamma$  is partially antipodal when  $K$  has fewer distinct eigenvalues than  $\Gamma$ . In particular, this is the case of antipodal distance-regular graphs ( $K$  with only two distinct eigenvalues), and the so-called half-antipodal distance-regular graphs ( $K$  with only one negative eigenvalue). We provide a characterization of partially antipodal distance-regular graphs (among regular graphs with  $d$  distinct eigenvalues) in terms of the spectrum and the mean number of vertices at maximal distance  $d$  from every vertex. This can be seen as a general version of the so-called spectral excess theorem, which allows us to characterize those distance-regular graphs which are half-antipodal, antipodal, bipartite, or with Kneser graph being strongly regular.

**Keywords:** Distance-regular graph; Kneser graph; Partial antipodality; Spectrum; Predis-  
tance polynomials.

**AMS subject classifications:** 05C50, 05E30.

## 1 Preliminaries

Let  $\Gamma$  be a distance-regular graph with adjacency matrix  $A$  and  $d+1$  distinct eigenvalues. In the recent work of Brouwer and the author [2], we studied the situation where the distance- $d$  graph  $\Gamma_d$  of  $\Gamma$ , or Kneser graph  $K$ , with adjacency matrix  $A_d = p_d(A)$ , has fewer distinct eigenvalues. In this case we say that  $\Gamma$  is *partially antipodal*. Examples are the so-called half antipodal ( $K$  with only one negative eigenvalue, up to multiplicity), and antipodal distance-regular graphs ( $K$  being disjoint copies of a complete graph). Here we generalize such a study to the case when  $\Gamma$  is a regular graph with  $d+1$  distinct eigenvalues. The main result of this paper is a characterization of partially antipodal distance-regular graphs, among regular graphs with  $d+1$  distinct eigenvalues, in terms of the spectrum

and the mean number of vertices at maximal distance  $d$  from every vertex. This can be seen as a general version of the so-called spectral excess theorem, and allows us to characterize those distance-regular graphs which are half antipodal, antipodal, bipartite, or with Kneser graph being strongly regular. Other related characterizations of some of these cases were given by the author in [8, 9, 10]. For background on distance-regular graphs and strongly regular graphs, we refer the reader to Brouwer, Cohen, and Neumaier [1], Brouwer and Haemers [3], and Van Dam, Koolen and Tanaka [6].

Let  $\Gamma$  be a regular (connected) graph with degree  $k$ ,  $n$  vertices, and spectrum  $\text{sp } \Gamma = \{\lambda_0^{m_0}, \lambda_1^{m_1}, \dots, \lambda_d^{m_d}\}$ , where  $\lambda_0(=k) > \lambda_1 > \dots > \lambda_d$ , and  $m_0 = 1$ . In this work, we use the following scalar product on the  $(d+1)$ -dimensional vector space of real polynomials modulo  $m(x) = \prod_{i=0}^d (x - \lambda_i)$ , that is, the minimal polynomial of  $A$ .

$$\langle p, q \rangle_\Gamma = \frac{1}{n} \text{tr}(p(A)q(A)) = \frac{1}{n} \sum_{i=0}^d m_i p(\lambda_i) q(\lambda_i), \quad p, q \in \mathbb{R}_d[x]/(m(x)). \quad (1)$$

This is a special case of the inner product of symmetric  $n \times n$  real matrices  $M, N$ , defined by  $\langle M, N \rangle = \frac{1}{n} \text{tr}(MN)$ . The *predistance polynomials*  $p_0, p_1, \dots, p_d$ , introduced by the author and Garriga [13], are a sequence of orthogonal polynomials with respect to the inner product (1), normalized in such a way that  $\|p_i\|_\Gamma^2 = p_i(k)$ . Then, it is known that  $\Gamma$  is distance-regular if and only if such polynomials satisfy  $p_i(A) = A_i$  (the adjacency matrix of the distance- $i$  graph  $\Gamma_i$ ) for  $i = 0, \dots, d$ , in which case they turn out to be the distance polynomials. In fact, we have the following strongest proposition, which is a combination of results in [14, 7].

**Proposition 1.** *A regular graph  $\Gamma$  as above is distance-regular if and only if there exists a polynomial  $p$  of degree  $d$  such that  $p(A) = A_d$ , in which case  $p = p_d$ .  $\square$*

Moreover, the Hoffman polynomial  $H$ , such that  $H(\lambda_i) = n\delta_{0i}$  and  $H(A) = J$ , turns out to be  $H = p_0 + p_1 + \dots + p_d$ . Also, as in the case of distance-regular graphs, the multiplicities of  $\Gamma$  can be obtained from the values of  $p_d$  since,

$$(-1)^i p_d(\lambda_i) \pi_i m_i = p_d(\lambda_0) \pi_0, \quad i = 1, \dots, d. \quad (2)$$

where  $\pi_i = \prod_{j \neq i} |\lambda_i - \lambda_j|$ . Indeed, let  $L_i(x) = \prod_{j \neq 0, i} (x - \lambda_j) / \prod_{j \neq 0, i} (\lambda_i - \lambda_j)$ . Then, since  $\text{dgr } L_i = d - 1$ , (2) follows from  $\langle L_i, p_d \rangle_\Gamma = 0$  for  $i = 1, \dots, d$ . Some interesting consequences of the above, together with other properties of the predistance polynomials are the following (for more details, see [4]):

- The values of  $p_d$  at  $\lambda_0, \lambda_1, \dots, \lambda_d$  alternate in sign.
- Using the values of  $p_d(\lambda_i)$ ,  $i = 0, \dots, d$ , given by (2), in the equality  $\|p_d\|_\Gamma^2 = p_d(\lambda_0)$ , and solving for  $p_d(\lambda_0)$  we get the so-called *spectral excess*

$$p_d(\lambda_0) = n \left( \frac{\pi_0^2}{m_i p_i^2} \right)^{-1}. \quad (3)$$

- For every  $i = 0, \dots, d$ , (any multiple of) the sum polynomial  $q_i = p_0 + \dots + p_i$  maximizes the quotient  $r(\lambda_0)/\|r\|_\Gamma$  among the polynomials  $r \in \mathbb{R}_i[x]$  (notice that  $q_i(\lambda_0)^2/\|q_i\|_\Gamma^2 = q_i(\lambda_0)$ ), and  $(1 =) q_0(\lambda_0) < q_1(\lambda_0) < \dots < q_d(\lambda_0) (= H(\lambda_0) = n)$ .

Let  $\Gamma$  have  $n$  vertices,  $d+1$  distinct eigenvalues, and diameter  $D(\leq d)$ . For  $i = 0, \dots, D$ , let  $k_i(u)$  be the number of vertices at distance  $i$  from vertex  $u$ . Let  $s_i(u) = k_0(u) + \dots + k_i(u)$ . Of course,  $s_0(u) = 1$  and  $s_D(u) = n$ . The following result can be seen as a version of the spectral excess theorem, due to Garriga and the author [13] (for short proofs, see Van Dam [5], and Fiol, Gago and Garriga [12]):

**Theorem 2.** *Let  $\Gamma$  be a regular graph with spectrum  $\text{sp } \Gamma = \{\lambda_0, \lambda_1^{m_1}, \dots, \lambda_d^{m_d}\}$ , where  $\lambda_0 > \lambda_1 > \dots > \lambda_d$ . Let  $\bar{s}_i = \frac{1}{n} \sum_{u \in V} s_i(u)$  be the average number of vertices at distance at most  $i$  from every vertex in  $\Gamma$ . Then, for any polynomial  $r \in \mathbb{R}_{d-1}[x]$  we have*

$$\frac{r(\lambda_0)^2}{\|r\|_\Gamma^2} \leq \bar{s}_{d-1}, \quad (4)$$

with equality if and only if  $\Gamma$  is distance-regular and  $r$  is a nonzero multiple of  $q_{d-1}$ .

*Proof.* Let  $S_{d-1} = I + A + \dots + A_{d-1}$ . As  $\text{dgr } r \leq d-1$ ,  $\langle r(A), J \rangle = \langle r(A), S_{d-1} \rangle$ . But  $\langle r(A), J \rangle = \langle r, H \rangle_\Gamma = r(\lambda_0)$ . Thus, Cauchy-Schwarz inequality gives

$$r^2(\lambda_0) \leq \|r(A)\|^2 \|S_{d-1}\|^2 = \|r\|_\Gamma^2 \bar{s}_{d-1},$$

whence (4) follows. Besides, in case of equality we have that  $r(A) = \alpha S_{d-1}$  for some nonzero constant  $\alpha$ . Hence, the polynomial  $p = H - (1/\alpha)r$  satisfies  $p(A) = J - S_{d-1} = A_d$  and, from Proposition 1,  $\Gamma$  is distance-regular,  $p = p_d$ , and  $r = \alpha q_{d-1}$ . The converse is clear from  $s_{d-1} = n - k_d = H(\lambda_0) - p_d(\lambda_0) = q_{d-1}(\lambda_0)$ .  $\square$

In fact, as it was shown in [11], the above result still holds if we change the arithmetic mean of the numbers  $s_{d-1}(u)$ ,  $u \in V$ , by its harmonic mean.

## 2 The results

As commented above, in [2] we studied the situation where the distance- $d$  graph  $\Gamma_d$ , of a distance-regular graph  $\Gamma$  with diameter  $d$ , has fewer distinct eigenvalues. Now, we are interested in the case when  $\Gamma$  is regular and with  $d+1$  distinct eigenvalues. In this context,  $p_d$  is the highest degree predistance polynomial and, as  $p_d(A)$  is not necessarily the distance- $d$  matrix  $A_d$  (usually not even a 0-1 matrix), we consider the distinct eigenvalues of  $p_d(A)$  vs. those of  $A$ . More precisely, given a set  $H \subset \{0, \dots, d\}$ , we give conditions for all  $p_d(\lambda_i)$  with  $i \in H$  taking the same value. Notice that, because the values of  $p_d$  at the mesh  $\lambda_0, \lambda_1, \dots, \lambda_d$  alternate in sign, the feasible sets  $H$  must have either even or odd numbers

### The case $\lambda_0 \notin H$

We first study the more common case when  $\lambda_0 \notin H$ . For  $i = 1, \dots, d$ , let  $\phi_i(x) = \prod_{j \neq 0, i} (x - \lambda_j)$ , and consider again the Lagrange interpolating polynomial  $L_i(x) = \phi_i(x)/\phi(\lambda_i)$ , satisfying  $L_i(\lambda_j) = \delta_{ij}$  for  $j \neq 0$ , and  $L_i(\lambda_0) = (-1)^{i+1} \frac{\pi_0}{\pi_i}$ , where  $\pi_i = |\phi_i(\lambda_i)|$ .

**Theorem 3.** *Let  $\Gamma$  be a regular graph with degree  $k$ ,  $n$  vertices, and spectrum  $\text{sp } \Gamma = \{\lambda_0, \lambda_1^{m_1}, \dots, \lambda_d^{m_d}\}$ , where  $\lambda_0 (= k) > \lambda_1 > \dots > \lambda_d$ . Let  $H \subset \{1, \dots, d\}$ . For every  $i = 0, \dots, d$ , let  $\pi_i = \prod_{j \neq i} |\lambda_i - \lambda_j|$ . Let  $\bar{k}_d = \frac{1}{n} \sum_{u \in V} k_d(u)$  be the average number of vertices at distance  $d$  from every vertex in  $\Gamma$ . Then,*

$$\bar{k}_d \leq \frac{n \sum_{i \in H} m_i}{\left( \sum_{i \in H} \frac{\pi_0}{\pi_i} \right)^2 + \sum_{i \notin H} \frac{\pi_0^2}{m_i \pi_i^2} \sum_{i \in H} m_i}, \quad (5)$$

and equality holds if and only if  $\Gamma$  is a distance-regular graph with constant  $P_{id} = p_d(\lambda_i)$  for every  $i \in H$ .

*Proof.* The clue is to apply Theorem 2 with a polynomial  $r \in \mathbb{R}_{d-1}[x]$  having the desired properties of  $q_{d-1}$ . To this end, first notice that, as  $q_{d-1} = H - p_d$ , we have  $q_{d-1}(\lambda_i) = -p_d(\lambda_i)$  for any  $i \neq 0$ . Thus, we take the polynomial  $r$  with values  $r(\lambda_i) = -t$  for  $i \in H$ , and  $r(\lambda_i) = -p_d(\lambda_i)$  for  $i \notin H, i \neq 0$ . Then, using (2),

$$\begin{aligned} r(x) &= -t \sum_{i \in H} L_i(x) - \sum_{i \notin H, i \neq 0} p_d(\lambda_i) L_i(x), \\ r(\lambda_0) &= -t \sum_{i \in H} (-1)^{i+1} \frac{\pi_0}{\pi_i} - \sum_{i \notin H, i \neq 0} p_d(\lambda_i) (-1)^{i+1} \frac{\pi_0}{\pi_i} \\ &= -t \sum_{i \in H} (-1)^{i+1} \frac{\pi_0}{\pi_i} - p_d(\lambda_0)^2 \sum_{i \notin H, i \neq 0} \frac{\pi_0^2}{m_i \pi_i^2}, \\ n \|r\|_\Gamma^2 &= r(\lambda_0)^2 + t^2 \sum_{i \in H} m_i + \sum_{i \notin H, i \neq 0} m_i p_d(\lambda_i)^2. \end{aligned}$$

Thus, (4) yields

$$\Phi(t) = \frac{r(\lambda_0)^2}{\|r\|_\Gamma^2} = \frac{n(\alpha t + \beta)^2}{(\alpha t + \beta)^2 + \sigma t^2 + \gamma} \leq \overline{s_{d-1}} \quad (6)$$

where

$$\alpha = \sum_{i \in H} (-1)^{i+1} \frac{\pi_0}{\pi_i}, \quad \beta = -p_d(\lambda_0) \sum_{i \notin H, i \neq 0} \frac{\pi_0^2}{m_i \pi_i^2}, \quad (7)$$

$$\gamma = \sum_{i \notin H, i \neq 0} m_i p_d(\lambda_i)^2 = \sum_{i \notin H, i \neq 0} \frac{p_d(\lambda_0)^2}{m_i} \frac{\pi_0^2}{\pi_i^2} = -p_d(\lambda_0) \beta, \quad \sigma = \sum_{i \in H} m_i. \quad (8)$$

Now, to have the best result in (6) (and since we are mostly interested in the case of equality), we have to find the maximum of the function  $\Phi(t)$ , which is attained at  $t_0 = \alpha\gamma/\beta\sigma$ . Then,

$$\Phi_{\max} = \Phi(t_0) = \frac{n(\alpha^2\gamma + \beta^2\sigma)}{\alpha^2\gamma + \beta^2\sigma + \gamma\sigma} \leq \overline{s_{d-1}} = n - \overline{k_d}.$$

Thus, using (7)-(8) and simplifying we get (5). In case of equality, we know, by Theorem 2, that  $\Gamma$  is distance-regular with  $r(x) = \alpha q_{d-1}(x)$  for some constant  $\alpha$ . If  $i \notin H, i \neq 0$ ,  $r(\lambda_i) = -p_d(\lambda_i) = \alpha q_{d-1}(\lambda_i) = -\alpha p_d(\lambda_i)$ , so that  $\alpha = 1$  since  $p_d(\lambda_i) \neq 0$ . Then, for every  $i \in H$ , we get

$$P_{id} = p_d(\lambda_i) = H(\lambda_i) - q_{d-1}(\lambda_i) = -r(\lambda_i) = t_0.$$

Conversely, if  $\Gamma$  is distance-regular, we have that  $\overline{k_d} = k_d$ , and, if  $P_{id}$  is a constant, say,  $\tau$  for every  $i \in H$ , we obtain, from (2), that  $\sigma = \frac{k_d}{p_d(\lambda_i)} \sum_{i \in H} (-1)^i \frac{\pi_0}{\pi_i} = \frac{k_d}{\tau} \alpha$ , whence  $\tau = \frac{1}{\sigma} k_d \alpha$ . Moreover,

$$nk_d = \|p_d\|_{\Gamma}^2 = \sum_{i \notin H} m_i p_d(\lambda_i)^2 + \sum_{i \in H} m_i \tau^2 = k_d^2 \sum_{i \notin H} \frac{\pi_0^2}{m_i \pi_i^2} + k_d^2 \frac{\left( \sum_{i \in H} \frac{\pi_0}{\pi_i} \right)^2}{\sum_{i \in H} m_i},$$

and equality in (10) holds. □

As mentioned above, when  $\Gamma$  is already a distance-regular graph, Brouwer and the author [2] gave parameter conditions for partial antipodality, and surveyed known examples. The different examples given here are withdrawn from such a paper.

**Example 4.** *The Odd graph  $O_5$ , on  $n = 126$  vertices, has intersection array  $\{5, 4, 4, 3; 1, 1, 2, 2\}$ , so that  $k_d = 60$ , and spectrum  $5^1, 3^{27}, 1^{42}, -2^{48}, -4^8$ . Then, with  $H = \{2, 4\}$ , the function  $\Phi(t)$  is depicted in Fig. 1. Its maximum is attained for  $t_0 = 6$ , and its value is  $\Phi(6) = 66 = s_{d-1}$ . Then,  $P_{24} = P_{44}$ .*

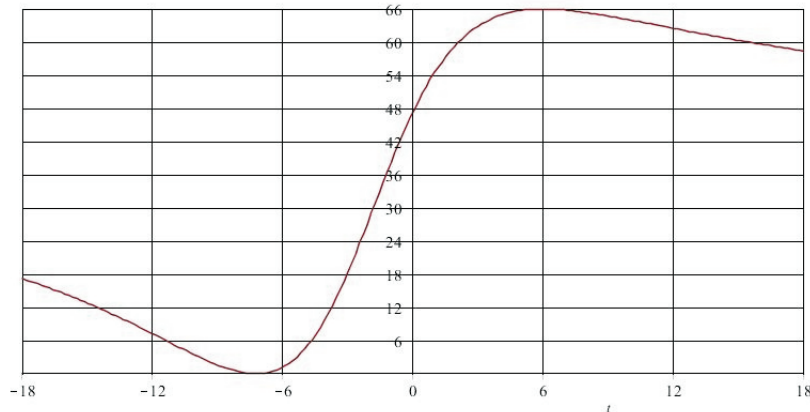


Figure 1: The function  $\Phi(t)$  for  $O_5$  with  $H = \{2, 4\}$ .

Notice that if, in the above result,  $H$  is a singleton, there is no restriction for the values of  $p_d$ , and then we get the so-called spectral excess theorem (originally proved by Garriga and the author [13]).

**Corollary 5** (The spectral excess theorem). *Let  $\Gamma$  be a regular graph with spectrum  $\text{sp } \Gamma$  and average number  $\overline{k_d}$  as above. Then  $\Gamma$  is distance-regular if and only if*

$$\overline{k_d} = p_d(\lambda_0) = n \left( \sum_{i=0}^d \frac{\pi_0^2}{m_i \pi_i^2} \right)^{-1}.$$

*Proof.* Take  $H = \{i\}$  for some  $i \neq 0$  in Theorem 3. □

As mentioned before, in [2] a distance-regular graph  $\Gamma$  was said to be half antipodal if the distance- $d$  graph has only one negative eigenvalue (i.e.,  $P_{id}$  is a constant for every  $i = 1, 3, \dots$ ). Then, a direct consequence of Theorem 3 by taking  $H = H_{\text{odd}} = \{1, 3, \dots\}$  is the following characterization of half antipodality.

**Corollary 6.** *Let  $\Gamma$  be a regular graph as above. Then,*

$$\overline{k_d} \leq \frac{n \sum_{i \text{ odd}} m_i}{\left( \sum_{i \text{ odd}} \frac{\pi_0}{\pi_i} \right)^2 + \sum_{i \text{ even}} \frac{\pi_0^2}{m_i \pi_i^2} \sum_{i \text{ odd}} m_i}, \quad (9)$$

*and equality holds if and only if  $\Gamma$  is a half antipodal distance-regular graph.* □

Recall that a regular graph is strongly regular if and only if it has at most three distinct eigenvalues (see e.g. [15]). Then, we can apply Theorem 3 with  $H_{\text{even}} = \{2, 4, \dots\}$  and  $H_{\text{odd}} = \{1, 3, \dots\}$  (and add up the two inequalities obtained) to obtain a characterization of those distance-regular graphs having strongly regular distance- $d$  graph.

**Corollary 7.** *Let  $\Gamma$  be a regular graph as above. Then,*

$$\overline{k_d} \leq \frac{n^2}{\left( \sum_{i \text{ even}} \frac{\pi_0}{\pi_i} \right)^2 + \left( \sum_{i \text{ odd}} \frac{\pi_0}{\pi_i} \right)^2 + \sum_{i \text{ even}} \frac{\pi_0^2}{m_i \pi_i^2} \sum_{i \text{ odd}} m_i + \sum_{i \text{ odd}} \frac{\pi_0^2}{m_i \pi_i^2} \sum_{i \text{ even}} m_i}, \quad (10)$$

*and equality holds if and only if  $\Gamma$  is a distance-regular graph with strongly regular distance- $d$  graph  $\Gamma_d$ .* □

**Example 8.** *The Wells graph, on  $n = 32$  vertices, has intersection array  $\{5, 4, 1, 1; 1, 1, 4, 5\}$  and spectrum  $5^1, \sqrt{5}^8, 1^{10}, -\sqrt{5}^8, -3^5$ . This graph is 2-antipodal, so that  $k_d = 1$ . Then, Fig. 2 shows the functions  $\Phi_0(t)$  with  $H_0 = \{2, 4\}$ , and  $\Phi_1(t)$  with  $H_1 = \{1, 3\}$ . Their (common) maximum value is attained for  $t_0 = 1$  and  $t_1 = -1$ , respectively, and it is  $\Phi_0(1) = \Phi_1(-1) = 31 = s_{d-1}$ . Then,  $P_{24} = P_{44}$  and  $P_{14} = P_{34}$ .*

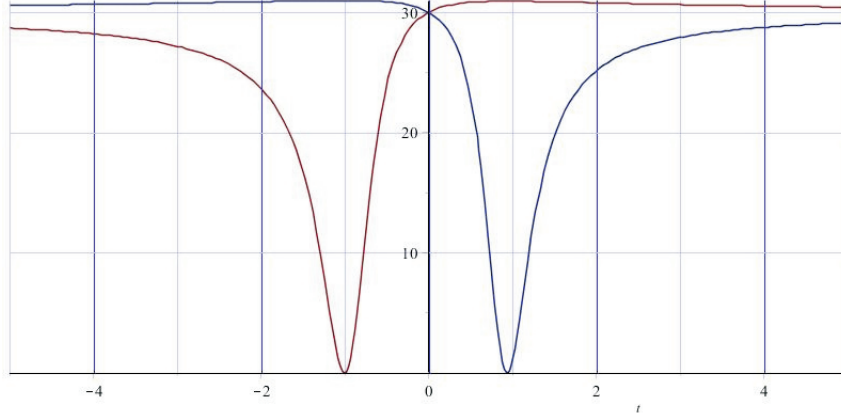


Figure 2: The functions  $\Phi_0(t)$  (in red) with  $H_0 = \{2, 4\}$ , and  $\Phi_1(t)$  (in blue) with  $H_1 = \{1, 3\}$  of the Wells graph.

In fact, the above expression can be simplified because  $\sum_{i \text{ even}} m_i + \sum_{i \text{ odd}} m_i = n$ ,  $\sum_{i \text{ even}} \frac{\pi_0}{\pi_i} = \sum_{i \text{ odd}} \frac{\pi_0}{\pi_i}$  (see [9]), and, from (3),  $\sum_{i \text{ even}} \frac{\pi_0^2}{m_i \pi_i^2} + \sum_{i \text{ odd}} \frac{\pi_0^2}{m_i \pi_i^2} = n/p_d(\lambda_0)$ . Anyway, we have written (10) as it is to emphasize the ‘symmetries’ between even and odd terms.

The following result was used in [2, 10] for the case of distance-regular graphs (where  $p_d(\lambda_i) = P_{id}$ ).

**Corollary 9.** *Let  $\Gamma$  be a regular graph with eigenvalues  $\lambda_0 > \lambda_1 > \dots > \lambda_d$ . Let  $H \subset \{1, \dots, d\}$ . Then,  $p_d(\lambda_i) = p_d(\lambda_j)$  for every  $i, j \in H$  if and only if  $\sum_{i \neq j} (m_i \pi_i - m_j \pi_j)^2 = 0$ .*

*Proof.* Notice that the right hand side of (5) is just the spectral excess  $p_d(\lambda_0)$ , which is given by (3). Then, the result follows from equating both expressions and simplifying.  $\square$

### The case $\lambda_0 \in H$

To deal with this case, we could proceed as above by defining conveniently a degree  $d - 1$  polynomial  $r$ . Then the proof is similar to the one for Theorem 3. If  $\lambda_0 \in H$  then  $p(\lambda_i) = p(\lambda_0)$  for any  $i \in H$ . Moreover, the odd indexes, cannot belong to  $H$ . In particular  $1 \notin H$ . For instance, a possible choice for  $r \in \mathbb{R}_{d-1}[x]$  is:

- $r(\lambda_0) = n - p_d(\lambda_0)$ ,  $r(\lambda_i) = -p_d(\lambda_0)$  for  $i \in H$ ,  $i \neq 0$ .
- $r(\lambda_i) = -tp_d(\lambda_i)$  for  $i \notin H$ ,  $i \neq 1$ ,

However, we can follow a more direct approach by using (6). First, the following result was proved in [2]:

**Proposition 10** ([2, Prop. 8]). *Let  $\Gamma$  be a distance regular graph with diameter  $d$ . If  $P_{0d} = P_{id}$  then  $i$  is even. Let  $i > 0$  be even. Then  $P_{0d} = P_{id}$  if and only if  $\Gamma$  is antipodal, or  $i = d$  and  $\Gamma$  is bipartite.*  $\square$

**Theorem 11.** *Let  $\Gamma$  be a regular graph with  $n$  vertices, spectrum  $\text{sp } \Gamma$  as above, and mean excess  $\overline{k_d}$ . Then, for every  $i = 1, \dots, d$ ,*

$$\overline{k_d} \leq \frac{n \left( m_i + \sum_{j \neq 0, i} \frac{\pi_0^2}{m_j \pi_j^2} \right)}{\left( \frac{\pi_0}{\pi_i} + \sum_{j \neq 0, i} \frac{\pi_0^2}{m_j \pi_j^2} \right)^2 + m_i + \sum_{j \neq 0, i} \frac{\pi_0^2}{m_j \pi_j^2}}. \quad (11)$$

Moreover:

- (a) *Equality holds for some  $i \neq d$  if and only if it holds for any  $i = 1, \dots, d$  and  $\Gamma$  is an antipodal distance-regular graph.*
- (b) *Equality holds only for  $i = d$  if and only if  $\Gamma$  is a bipartite, but not antipodal, distance-regular graph.*

*Proof.* The inequality (11) follows from (6) by taking  $H = \{i\}$  for some even  $i \neq 0$ , and choosing  $t = p_d(\lambda_0)$ . Then, in case of equality, Theorem 3 tells us that  $\Gamma$  is distance-regular. Then,  $\Gamma_d$  is a regular graph with equal eigenvalues  $P_{0d}$  and  $P_{id}$ . So, the result follows from Proposition 10.  $\square$

**Example 12.** *For the Wells graph the right hand expression of (11) gives  $1 (= k_4)$  for any  $i = 1, \dots, 4$ , in concordance with its antipodal character. In contrast, the folded 10-cube  $FQ_{10}$ , on  $n = 512$  vertices, has intersection array  $\{10, 9, 8, 7, 6; 1, 2, 3, 4, 10\}$  and spectrum  $10^1, 6^{45}, 2^{210}, -2^{210}, -6^{45}, -10^1$ . Then, the right hand expression of (11) gives 234.16, 293.36, 293.36, 234.16 for  $i = 1, 2, 3, 4$ , respectively, and  $126 (= k_5)$  for  $i = 5$ , showing that  $FQ_{10}$  is a bipartite distance-regular graph, but not antipodal.*

Another characterization of antipodal distance-regular graphs was given by the author in [8] by assuming that the distance  $d$ -graph of a regular graph is already antipodal.

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